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THE CENTER OF JORDAN DOMAINS

AKIO OSADA

1. Introduction

An extremal problem will be considered and treated in this paper. Given a Jordan domain G in the complex plane,a continuous function F(w, z)on the product space $\overline{G}x\overline{G}$ and a positive Borel mesure dm(Z) with total mass 1 on the boundary dG, we shall define the variance V(F(Z, z), dm(Z)) of F(Z, z) at z with respect to dm(Z) for any z in G and consider the problem of minimizing it. We are also interested in how the minimizing points may be characterised in connection with the Riemann mapping function. The result of this paper, however, is to give a partial answer for this.

2. Examples.

We shall define the variance of F(Z, z) at z with respect to dm(Z) by

$$\int_{dG} \{F(Z,z)\}^2 \, dm(Z) - \{\int_{dG} F(Z, z) dm(Z)\}^2,$$

and give two examples. First, as a Jordan domain G, a continuous function F(w, z) and a positive Borel measure dm(Z) stated in 1, we take the unit disk D : $|z| < 1$,the function

$$-\log |w-z|$$

and the measure $(2\pi)^{-1}dt$ on dD : $|z| = 1$ respectively. Then,by definition, the variance of $-\log|Z-z|$ is

$$(2\pi)^{-1}\int_0^{2\pi}\{\log|Z-z|\}^2\,dt-(2\pi)^{-2}\{\int_0^{2\pi}\log|Z-z|\,dt\}^2,$$

where $Z=\exp(it)$. Since the potential

$$-\int_0^{2\pi}\log|Z-z|\,dt$$

vanishes for any z in D ([1]), we can easily conclude that the variance

$$V(-\log|Z-z|,\ (2\pi)^{-1}dt,\ z)$$

of $-\log|Z-z|$ at z with respect to $(2\pi)^{-1}dt$ attains its minimum if and only if $z=0$.

The second example is given, using $|w-z|^2$ instead of $-\log|w-z|$. Then, by elementary calculations, we see that the variance

$$V(|Z-z|^2,\ (2\pi)^{-1}dt,\ z)$$

is equal to $2|z|^2$. Consequently, we also see that the variance above attains its minimum if and only if $z=0$. At this stage, however,we can not show whether the minimizing point should be unique under more general conditions. All what we can prove here is some property concerning the kernel.

3. The theorem

Theorem. Let $f(x)$ with $f(1)=0$ be real-valued and analytic on the interval $(0, 2)$, possessing Maclaurin expansion

$$\sum_{n=1}^{\infty} a_n (x-1)^n \text{ at } x=1.$$

Then, a necessary and sufficient condition that the function

$$u(z)=(2\pi)^{-1}\int_0^{2\pi} f(|Z-z|^2)\,dt \quad (Z=\exp(it))$$

be harmonic in D is that

$$\Sigma A(m, n-m)\,b(2m)+\Sigma B(m, n-m)\,b(2m+1)=0$$

for each $n>1$ where the first summation or the second is taken from $m=[n/2+1]$ to $m=n$ or $m=[(n+1)/2]$ to $m=n$ respectively and further

$b(k) = ka_k + (k+1)a_{k+1}$,

$A(m, n-m) = (2m)! / \{(2n-2m+1)[(2m-n-1)!]^2\}$ and $B(m, n-m) = (2m+1)! / \{(2n-2m)! [(2m-n)!]^2\}$.

Proof. We have only to see the necessity. Noting the equality

$$|Z-z|^2 = 1 + r^2 - 2r\cos(s-t)$$

where $Z = \exp(is)$ and $z = r\exp(it)$, we set

$$f(|Z-z|^2) = v(r, s, t).$$

Then, we have

$$\Delta u = (2\pi)^{-1} \int_0^{2\pi} \Delta v ds,$$

which is an immediate consequence of the analyticity of $f(x)$. Moreover, since

$$\Delta v = \Delta 4 \{f''(x)x + f'(x)\}$$

where $x = 1 + r^2 - 2r\cos(s-t)$, the sufficiency amounts to the proof for the equality

(1) $$\int_0^{2\pi} \{f''(x)x + f'(x)\} ds = 0$$

for any x $(0 < x < 2)$. The integrand

$$f''(x)x + f'(x),$$

however, is a periodical function of s with period 2π and hence, the left hand side of (1) is equal to

$$\int_t^{t+2x} \{f''(x)x + f'(x)\} ds,$$

which in turn changes into the integral

$$2\int_0^{\pi} g(1 + r^2 - 2r\cos(s))ds$$

where $g(x) = f''(x)x + f'(x)$, by virtue of the property that the integrand above is an even function of s. Therefore, all what we can use is the equality

(2) $$\int_0^{\pi} \{f''(x)x + f'(x)\} ds = 0$$

where $x=1+r^2-2r\cos(s)$, for any $r(0<r<1)$. Since the function x is increasing on $[0,\pi]$ for a fixed r, we can consider the inverse

$$s=\cos^{-1}\{(1+r^2-x)/(2r)\},$$

the derivative $(ds)/(dx)$ of which is easily seen to be

$$(4r^2)^{-1}\{(1+r)^2-x\}\ \{x-(1-r)^2\}.$$

Consequently, by virtue of this derivative, (2) can be rewritten as

(3) $$\int_a^b \{f'(x)x\}'\{[x-a]\ [b-x]\}^{-1/2}\,dx=0$$

where $a=(1-r)^2$ and $b=(1+r)^2$, for $0<r<1$. Using the expansion

$$f(x)=\Sigma a_n(x-1)^n$$

and setting $x-1=(2r-r^2)t$, we can find that (3) amounts to the equality

(4) $$\Sigma nb(n)\ (2r-r^2)^{n-1}\int_{-1}^{c} t^{n-1}\{(1+t)(c-t)\}^{-1/2}\,dt=0$$

where $c=(2r+r^2)/(2r-r^2)$, for any $r(0<r<1)$. The last stage of this calculation is to define the function $y=y(t)$ by the relation

$$t=(c-1)/2+(c+1)/2\cos y.$$

Then, (4) changes into the form

$$\Sigma nb(n)\int_0^\pi (r^2+2r\cos y)^{n-1}\,dy=0$$

for any $r(0<r<1)$. Here we should expand the left hand side above as an infinite series of r. To this end, by virtue of elementary calculations, we note that

$$\int_0^\pi (r^2+2r\cos y)^{n-1}\,dy$$

is equal to, if $n=2m$,

$$\pi\sum_{p=0}^{m-1}(2m-1)!/\{(2p+1)!\ [(m-p-1)!]\}\ r^{2m+2p}$$

and if $n=2m+1$,

$$\pi\sum_{p=0}^{m}(2m)!/\{(2p)!\ [m-p]!]\}\ r^{2m+2p}.$$

Consequently, the coefficicient of r^{2n} in the expansion mentioned above becomes the form stated in the theorem. Thus our proof is complete.

REFERENCES

[1] M.Tsuji. Potential Theory in Modern Function Theory. Maruzen;Tokyo,1959.